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Complexity Economics and the Accounting Framework

by

Frederico Botafogo

Abstract

This working paper introduces a formal language to frame the concept of complexity within economic theory. The purpose is to provide a consistent analytical framework within which the varied aspects of complexity may be given expression.

The intuition underlying the framework is provided by double-entry bookkeeping and the accounting equation whereby assets equal liabilities for all possible agents in the economic system. The discussion is focused on foundational issues; markets, preferences, convexity in production, general equilibrium are not deemed to be required in describing an economic system. The basis for developing the framework is given solely by accounting theory. Linear algebra, in particular vector and tensor analyses, provide the mathematical elements required to construct said framework.

Complexity in economics is understood to imply subjective beliefs by interacting agents, who chose strategies given their particular goals, and further who update said strategies as time goes by. The resulting framework is, accordingly, dynamic and inherently uncertain. Therein, the accounting equation is seen as the expression of a solution concept, as this is defined in Game Theory. And economic value is a measurement scale which informs how distinct resource subsets relate to each other as input to output given a strategic economic process consisting of production, trade, or both.

The epistemological background which supports the framework relies on the representational theories of measurement.

The paper is organised as a list of talking points. Three appendices cover the methodological approach being used, the mathematics knowledge being required, and a short review of what the representational approach to measurement entails.

Keywords

‘Complexity Economics’; ‘Complexity Accounting’, ‘Double-Entry Bookkeeping’, ‘Value and Income Measurements’.

Author(s) details

Frederico Botafogo – Southern Institute of Technology, New Zealand

fredericobotafogo@gmail.com

Complexity Economics and the Accounting Framework

The conference title refers to 'complexities', in the plural; further, the conference background statement clearly allows for different, and possibly conflicting, approaches to the term 'complexity' and how it would apply to economics. Thus, the first issue I address in this working paper concerns the scope of the forthcoming discussion on complexity and economics.

Arthur (2013) provides a particular blueprint for Complexity Economics. He explains how complexity[1] enters into economics—point (2) in the conference background statement—by means of two ideas:

> *Agents thus live in a world where their beliefs and strategies are constantly being "tested" for survival within an outcome or "ecology" these beliefs and strategies together create.*
>
> *In this framework time, in the sense of real historical time, becomes important, and a solution is no longer necessarily a set of mathematical conditions but a pattern, a set of emergent phenomena, a set of changes that may induce further changes, a set of existing entities creating novel entities.*

I concur with Arthur (2013) that incorporating these two ideas yields a different framework for economic thought. This working paper is focused on discussing a framework for economic analyses such that the ideas of subjective beliefs, interacting strategies, and a dynamic approach to processes may be properly addressed *within* the formal language being defined by the framework. The focus is on the formal language with which to frame economic reality.

Fullbrook (2013) provides a list of ten points to define what he calls the New Paradigm in Economics (NPE), said list summarising the varied contributions by those economists who have published papers in the WEA journal. The two points most relevant for the forthcoming discussion are points 3 and 7:

> *The NPE chooses its math, as in both classical and modern physics, on the basis of its isomorphism to real-world phenomena, including construction of real-world empirical models using real data (i.e., prioritises the empirical over apriorism).*

This is point 3. The key term here is 'isomorphism', meaning that the mathematical structure being chosen to describe and analyse some underlying reality must reflect the structure of the data pertaining to that underlying reality. This point further clarifies the focus of this working paper: it is about choosing the math, that is, the proper formal language with which to model economic processes based on subjective beliefs and interacting strategies, which both must be modelled from a dynamic perspective. Further, the idea of 'isomorphism' is reviewed using the epistemological approach of Representational Theories of Measurement (RTM)[2].

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[1] Note the singular: Arthur being associated with the Sante Fe Institute, he coined 'Complexity Economics' by reference to mathematics, particularly Chaos Theory.

[2] Roberts (1984) provides a good introduction to RTM.

The NPE regards agents as social beings, recognises emergent properties and structures as fundamental to economic reality and thereby the need for a multidimensional ontology.

This is point 7. The key terms are 'social beings' which are defined by properties that 'emerge' within a system of economic interactions. This adds clarity to the already mentioned focus of this paper: it is about defining, within the formal language, what the term 'agent' is supposed to convey and how 'agent' relates to beliefs and strategies. Further, it is about providing that language with sufficient flexibility for the introduction of 'novel entities', which would arise out of actual or perceived changes occurring over time.

Points 3 and 7 provide a backdrop for the aim of this paper, which is to contribute to the development of Fullbrook's NPE.

But the problem with heterodox economics has always been that it could never challenge conventional economics because it is a fragmented collection of critiques and insights which lacks any analytical unity. It is here that the theory of complex adaptive systems can make a real contribution by providing a consistent analytical framework within which many of the best contributions of heterodox economists can be given expression. (Foster, 2011)

For a full appreciation of the foregoing point, consider the conventional economics' formal framework. The prevailing framework was *properly* formalised for the first time by Debreu (1959). Some heterodox economists criticise this framework for being unrealistic. To be clear about the scope of this working paper, note that this kind of criticism is not relevant or applicable. As Fullbrook (2012) argues, paraphrasing Einstein,

Whether you can observe a thing or not depends on the theory which you use. It is theory which decides what can be observed.

My point is that the concepts of 'agent', 'strategy', 'beliefs' and 'uncertainty', and mostly 'dynamics', are all modelled either as a natural derivation of the formal elements in Debreu's axiomatic system or in direct opposition to them. It is always by reference to Debreu' system. A good example is given by Debreu himself,

Allegiance to rigor dictates the axiomatic form of the analysis where the theory, in the strict sense, is logically entirely disconnected from its interpretations. […] such a dichotomy … makes possible immediate extensions of that analysis without modification of the theory by simple reinterpretations of concepts; this is repeatedly illustrated below, most strikingly perhaps by Chapter 7 on uncertainty. (Debreu, 1959: preface)

Debreu deals with uncertainty within a formal framework that is inherently deterministic. This is not an oxymoron, however. This is possible because it holds at the semantic level, as opposed to the syntactic level, when concepts are reinterpreted such that they are made to relate, sometimes more realistically, to the underlying reality that the economist is trying to model.

I reinforce the point by noting, anecdotally, that every time I suggested to an economist—who, by the way, could be affiliated with any school of economic thought—the possibility of modelling an economic system without reference to the concept of 'market', the reaction was always negative, ranging from their ridiculing the suggestion to dismal or open aggressiveness. Yet, there is no such thing as a 'market' for apples, according to Debreu's definition. Consumers buy apples in the supermarket, which is not a producer but an intermediary. This changes how equilibrium is supposed to work. Further, there are different kinds of apples, say green or red apples, a fact that begs the question: how is one to define the market for apples?

I what follows I suggest to start afresh: no *a priori* assumptions about markets, consumers having preferences, or firms being formally framed by production functions. Most importantly, no money; there is no socially constructed currency being introduced in the framework to facilitate trade or the storage of value. I concur with Fullbrook (2002) that money should be framed as a measure of exchange-value, which in turn would satisfy the axioms of a Boolean algebra. Thus, the concept of value must be made clear, first, before money may be addressed. Similarly to Debreu's (1959) axiomatic system, I do not account for money (yet).

Where to start this paradigmatic new formal framework afresh? As with any axiomatic construction, a set of intuitive, self-evident propositions is required which are deemed applicable to the topic of interest at hand, herein the economic reality. To do so, I turn to accounting, specifically to double-entry bookkeeping (DEB). All economic data, be it the costs and prices determined by individual firms at the micro-level or the aggregated income measures determined by governments and central banks at the macro-level, is *framed* by the logic of double-entry bookkeeping. Some active contributors to the WEA journals (e.g., Steve Keen) base their approach to the nature of money on DEB (i.e., the circuitist view of fiat money). Further, taking a historical approach, DEB was introduced many centuries before economics was recognised as a subject of scholarly inquiries. Some historical economists and sociologists, Sombart and Weber, even suggest a relationship between the rise of capitalism and the logic of DEB (Yamey, 1949). Finally, despite being around for over 500 years, the mathematical structure that supports DEB has not changed. This suggests DEB is a fundamental feature of past and present capitalistic economies.

> *[…]economists really [know] only five things - (1) the national accounts add up, national product equalling national income; (2) the balance of foreign payments adds up; (3) the money supply is 'created' by a system of banks in which each holds as a reserve only a fraction of the money deposited with it; and a couple of demographic truths, which might be illustrated by the growth of the unmarried population by exactly two when a husband and wife get a divorce. Learning to think like an economist consists in good part of learning to speak such bits of accounting logic. As Adam Smith said in the first sentence of* An Enquiry into the Nature and Causes of the Wealth of Nations, *affirming the truth that national income equals national product,* 'The annual labour of every nation is the fund which originally supplies it with all the necessaries and conveniences of life which it annually consumes.' *In view of its importance in their work the economists could be expected to have an interest in accounting. Once they did. But now they don't. (Klamer & McCloskey, 1992)*

The rest of this paper is organised as a list of talking points. I consider this structure to be the most appropriate for this conference given both the wide scope of possible topics as well as the clear exploratory nature of the debate to take place. Further, I do not know what to expect from the readership; thus, I reckon, let a list of ideas be presented and let wait and see what the reactions by the conference participants will be. Should a conversation take place, I will then gather elements, as the conversation proceeds, to focus on how best to order and present the ideas herein. Hopefully, the outcome will be a paper structured along the usual lines.

A list of talking points

1st idea: To cross a bridge, one can go from A to B or alternatively from B to A.

The prevailing approach to complexity, supported by a vast literature, is that knowledge gained and developed within the mathematical field of complexity (e.g., dynamic systems, chaos theory, etc.) is beneficial to gaining and developing knowledge in the field of economics. There seems to be a preferred direction when trying to bridge the two fields, namely starting with the mathematics of complexity (its ideas and formalism), then proceeding to frame economics accordingly. The focus being on the maths, the foundational concepts of economics are not reviewed nor changed to fit the new properties of the mathematical framework. If an agent is seen to be a person or a firm, with a given identity, then no matter the sophistication of the mathematics being used, that agent will not evolve (i.e., change its identity) in view of its interactions with other agents. Thus, there is only so much that can be gained from taking *just* this approach. This has been recognised before, say by Gallegati et al. (2006), who question how income may be generated within a framework wherein only conservative transformations take place.

I develop a formalism to frame accounting and its supporting procedure of double-entry bookkeeping (DEB). I proceed with an axiomatic construction such that DEB is the axiomatically true starting point. Specifically, I assume that value and cost are conserved whenever a transformation takes place. I choose the proper mathematical objects accordingly. Only after the framework is in place, do I consider tackling any economic conceptual issue.

2nd idea: Paradoxes arise out of a limitation in the language being used.

Before addressing accounting and DEB, let it be noted that a motivation for this conference is the realisation that the 'usual' approach to mathematically framing economics (i.e., the preferred formalism of neoclassical economics) is prone to experimental refutation. There may be many reasons for such an outcome. However, the focus of this working paper is on the formalism' structure, that is, on its axioms.

I wish to make an explicit reference to paradoxes which, in my understanding, arise from shortcomings in the language being used, be it a natural language or, as is the case here, the formal language with which economics is framed and conveyed.

Solving a paradox requires re-conceptualising one or more concepts within a single framework of analysis. An example I like is Zeno's paradoxes about movement (e.g., the race between Achilles and the turtle), which were solved only in the nineteenth century when Cantor, using axiomatic set theory, rigorously framed the concept of infinity. The point is that Cantor' solution contradicts the theretofore evident understanding encapsulated within Euclid's fifth axiom that the whole is greater than its parts. I claim an alternative axiomatization for the decision making process is possible such that it does not rely on the concept of utility.

3rd idea: At its most intuitive level, utility is applicable to consumers.

A good axiomatic system, being consistent, cannot be invalidated by logic. To break away one must change one or more of its axioms. A key feature of an axiom is that it must be intuitively evident. Within the present context, this means changing the focus of the decision making process from consumers who have preferences towards 'entities' (e.g., people as well as any kind of organisation) that have purposes or goals. Although preferring an apple to an orange characterises a consumption goal, not all goals can be characterised by means of preferences in consumption, at least not in any straightforward way (this is achieved in neoclassical economics by reference to general equilibrium only).

I am changing the axiomatic system wherein decisions are modelled. The change is based on approaching the concept of value in a way different from neoclassical economics' way. Rather than considering value as an inherent attribute of commodities, value is seen as purely relational. At the most basic level, resources are classified as inputs or outputs in view of a system of productive transformations, which may include consumption, taking place over a time interval. The input-output relationship is what defines value as an attribute of resources.

The intuition of relational attributes is not strange to economists. Since Adam Smith posited that commodities have *value in use* (an inherent quality) and *value in trade* (a systemic quality), the idea of value as a relational attribute has been well established. The neoclassical theory of value combines the measurement of preferences that are inherent to consumers with the systemic requirement that demand be equal to supply.

If I may *fast-forward* my argument with the view of providing some intuition about where this is leading us, the consequence of dealing with a purely relational attribute is that the logic of economic decisions ends up-side-down when compared to the usual one. Figure 1 illustrates both logical approaches, which assume that economic decisions are predicated on an analysis of value being given by a price or a cost. The usual, bottom-up approach is depicted on the left side of Figure 1. Prices are inherent to resources and are assumed as given before a decision can be made. Given prices, costs are constructed by adding up prices of all inputs making up a particular output. The difference between the output price and its costs yields a measure of income. The possibility of using the same inputs to obtain different outputs yields different income possibilities. Comparing any two such possibilities yields a measure of performance.

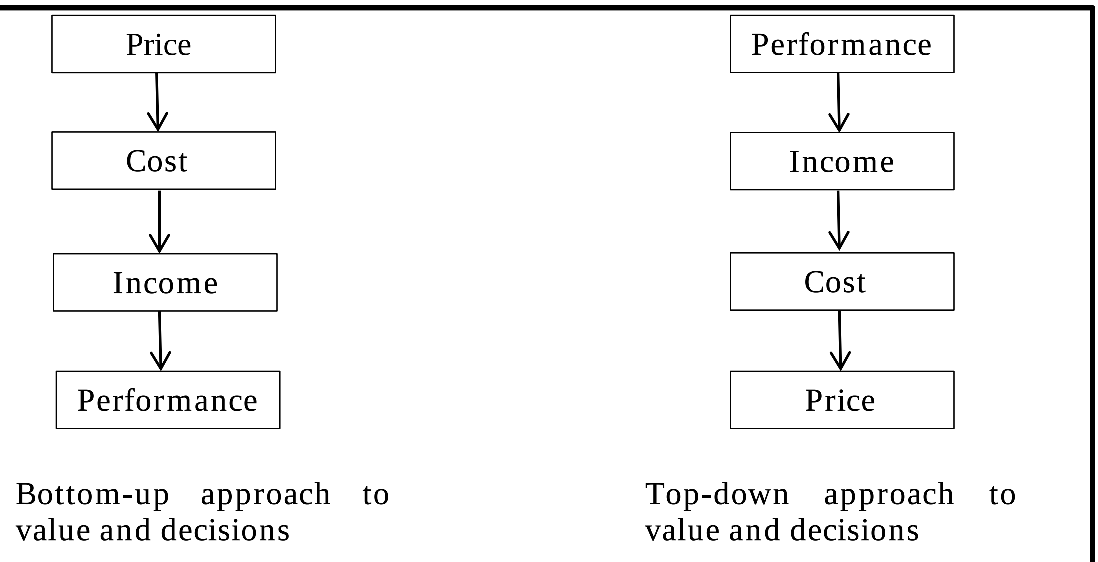

Figure 1: comparing two logical approaches to value and decisions

The top-down approach depicted in the right side of Figure 1 requires decision-makers having goals. At the axiomatic level, this entails real goals not being expressed in monetary terms. They consider alternative strategies and choose one they perceive will perform better. A structural condition is then imposed on the chosen strategy: it represents a physical process over a given time interval whereby initial resources are matched to final outcomes that express the desired goals. By means of this condition the mathematical representation of a strategy is assigned an income measure. Once that measure is available, costs obtain as derived measurements and from them prices are subsequently constructed.

4th idea: *To clarify the foregoing, Representational Theories of Measurement (RTM) are introduced. Within RTM, a scale is a morphism mapping an empirical structure to a representational structure which, here, is framed as a vector space.*

I axiomatically require the framework representing the measurement of income and economic value to be a linear vector space. Therein, both addition and multiplication have a role to play:

> *"It is widely agreed that physical quantities combine additively within a single dimension (at least when that dimension is extensively measurable) and that different dimensions combine multiplicatively; that the multiplicative structure is very much like a finite-dimensional vector space over the rational numbers; that the existence of basic sets of dimensions in terms of which we can express the remaining dimensions corresponds to the existence of finite bases in the vector space; and that numerical physical laws are almost always formulated in terms of a very special class of functions defined on this space."* (Krantz, Luce, Suppes, & Tversky, 1971: p.459)

Measurement theory requires an underlying relational system that embodies empirical observations, say (A, R, ∘), which is then mapped by a homomorphism to a representational, numerical system. Refer to Appendix 3 for details on this particular view of epistemology.

The Underlying Set A:

Here, the underlying set A contains pairs of inputs to outputs. These pairs have dimensions since they express a rate of conversion; they are dimensional ratios. Their units are given by the inverse of the input quantity to the output quantity. When trading an apple for an orange, the unit of the ratio is given by units of oranges divided by units of apples. The ratio is denoted $x^a y_o$—x standing for the input and y for the output. When several inputs convert to several outputs, the 'generalised' ratio is represented by the operator **|x)(y|** (i.e., a matrix) such that **(x||x)(y|** = **(y|** conveys that the vector of inputs **(x|** is converted to the vector of outputs **(y|**. Refer to Appendix 2 for details on this particular notation.

The important point being made is that the set of quantities (i.e., the commodities) is *not* the underlying set. I am focusing on value measurement and value does not arise from quantity, although quantity may have an impact on value. In what follows, value is process-oriented.

The binary operation ∘ (a.k.a. concatenation):

The symbol ∘ stands for an operation that express the ability to combine the elements in A. Since the $x^i y_j$ ratios have dimensions (*i* indexes the inputs while *j* indexes the outputs), they combine in a vector-like way and yield the tensor **|x)(y|**. An equation is posited as an accounting law in the representational vector space, namely **(x||x)(y||y)** = 1, such that it converts dimensional numbers in pure numbers. This law is a representational statement that reads as 'inputs yield outputs' and it does not rely on the physical attributes of resources, although it must be consistent with the perceived physical process taking place (e.g., production).

In this expression, **(x|** represents a vector of physical input resources, **|x)(y|** represents a generalised converting ratio, and **|y)** is not a physical vector but rather a vector expressing the desired output resources expected at the conclusion of the underlying process, said process henceforth to be referred to as a strategy. Specifically, **|y)**'s dimensional components are the values being assigned to the output resource bundle.

The value of a bundle is given by the several quantities within the bundle multiplied by the particular values of each of its components. However, the whole bundle value is axiomatically set equal to 1—this makes the definition of value consistent with Fullbrook's (2002) requirements. For a bundle consisting of input resources **(x|** being assigned the input values **|x)**, the value requirement yields **(x|x)** = 1. For outputs, it reads **(y|y)** = 1. Thus, the bundle's total value is constant over the duration of a strategy, herein being normalised to 1. Further, the statement **(x||x)(y| = (y|** conveys that earlier inputs are converted to later outputs while the symmetric statement **|x) = |x)(y||y)** conveys that later values determine earlier values in a similar way to present value arising out of a discounted cash-flow.

The weak order R:

The extant approaches to accounting measurement are predicated on the view that the relation R is a strict simple order (e.g., Willett, 1991: 'cost structures'; Vickrey, 1970: p.737). I suggest going counter to this generally accepted view.

It is true that costs, prices, or values are commensurable (i.e., being bigger, smaller, or equal to one another) and this is the key to making sense of commodities' economic and accounting values. Further, a fundamental ratio scale seems therefore natural. However, this is now challenged by the argument that ratio scales can be *derived* from weak order relations.

It is possible to derive a ratio scale when the relation R is a strict weak order (Roberts, 1984: 122-131). Accordingly, the observable input-output relation is now framed by reference to a weak order and not by means of a strict simple order, as usually done.

Recall that consumers' preferences are framed as a strict simple order. A strict simple order satisfies three properties, namely transitivity, asymmetry, and completeness. In contrast, a strict weak order does not satisfy completeness. It satisfies only two properties, asymmetry and negative transitivity. For all *a* and *b* in A, the former means *not a*R*a* (anti-reflexivity) and *if a*R*b*, *not b*R*a* (anti-symmetry) while the latter means *if not a*R*b* and *not b*R*c*, then *not a*R*c*, for all *a*, *b*, *c* in A.

The intuition that supports the axiom of weak order is based on the economic setting wherein only barter transactions take place (i.e., the pure exchange economy). Consider someone's trading an apple for an orange. They relate as input to output and the following holds true: Apple is the input; therefore, it is not the output; the relation is anti-reflexive. Further, orange is not the input to apple; the relation is anti-symmetric. Finally, if someone cannot trade apples for bananas and neither bananas for carrots, one cannot get hold of carrots by this trading sequence (apples, bananas, carrots). One still may end up with carrots but only if another applicable sequence is available. If not, then at the fundamental level a set of incommensurable resources may exist. However, this does not imply that all resources are representationally incommensurable. The weak order states a weaker condition on the underlying order such that, depending on the availability of trading sequences, any resource may be comparable to, even if not tradable for, any other.

It is possible to organise input-output relations in several time sequences such that a net of processes obtain and such that, if the sequences are long enough, the net makes all resources commensurate. Further, when purposes are attached to processes such that they may be called strategies, a valuation procedure is possible such that values are determined in terms of the trading ratios that are set by social agreement.

To insist: the point is that once the set of quantities (i.e., commodities) no longer is seen as the underlying set and, further, the concept of value is constructed based on barters sequences that yield strategies, the focus of the fundamental measurement analysis must be shifted towards the elementary components in a strategy, namely the input to output relationships. However, to complete this measurement analysis, additional structure is now required.

5th idea: *To construct accounting measurement, reference is made to partitions. The set of inputs-outputs pairs is partitioned such that resources are classified (i.e., given names) accordingly. A probabilistic value measurement obtains.*

Consider that the concept of strategy is teleological in nature. When a strategy is conceived, reference is made to a goal or purpose. With this in mind, consider the case of an egg that is broken in two. Once broken, it loses its substance and hatching chicks is no longer possible. Thus, if the egg has value because by its hatching it gives place to a chicken, a chicken which in turn lays more eggs, some of which will be eventually consumed, then from a strategy perspective the egg is indivisible because its breaking destroys the strategy. However, consider that the claim to the egg is always divisible, as when two persons own it jointly. The chick that eventually hatches may also be claimed jointly, as when the owners' share the future stream of eggs that the grown up chicken lays over its life.

Physical conditions empirically limit the ability to separate processes. The measurement of value in the form envisaged here requires an axiom to separate processes beyond physical limits, the new limit to be determined by social convention. Before this is discussed, some additional background information is brought to the fore.

The Dual Logic of Partitions:

Ellerman (2014b) introduces the dual[3] logic of partitions, a logic that models a universal set with indefinite elements becoming more definite as distinctions are made. A naïve rendering of Ellerman's dual logic of partitions follows.

> *[…] the common-sense view of reality […] is expressed at the logical level in Boolean subset logic. Each element in the Boolean universe set is either definitely in or definitely not in a subset, i.e., each element either definitely has or does not have a property. Each element is characterized by a full set of properties, a view that might be referred to as "definite properties all the way down."* (*Ellerman, 2013: p.3*)

> *We can now describe how the dual logic of partitions captures at the logical level a vision of reality with objectively indefinite (or indistinct) entities. The key step is to interpret a subset S as a single objectively indefinite element.* (*Ellerman, 2013: p.11*)

Meta-physical notions of substance and form illustrate the conceptual duality of partitioning in terms of the two lattices in Figure 2, adapted from Ellerman (2013).

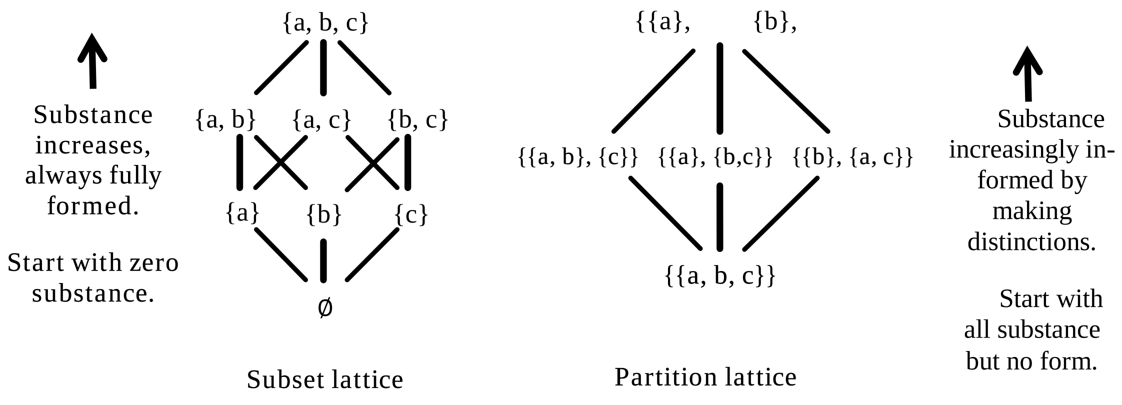

Figure 2: Conceptual duality between the Subset and the Partition lattices (adapted from Ellerman, 2013)

> *The progress from bottom to top of the two lattices could also be described as two creation stories. Subset creation story: "In the Beginning was the Void", and then elements are created, fully propertied and distinguished from one another, until finally reaching all the elements of the universe set U. Partition creation*

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[3] The term dual refers to categorical logic: "*Partitions on a universe set are dual to subsets of a universe set in the sense of the reverse-the-arrows category-theoretic duality–which is reflected in the duality between quotient objects and subobjects throughout algebra.*" (Ellerman, 2014b)

*story: "In the Beginning was the Blob", which is an undifferentiated "substance," and then there is a "Big Bang" where elements are created by the substance being objectively in-formed by the making of distinctions (e.g., breaking symmetries) until the result is finally the singletons which designate the elements of the universe U. (*Ellerman, 2013: p.14*)*

The 5th idea requires that physical resources be classified (i.e., be given names) by reference to strategies. The universal set **S** consists of strategies, teleological processes connecting aggregated inputs to aggregated outputs. By partitioning **S**, a classification scheme obtains such that the distinct input-output relationships *in-form* the elements in the resource set (see next subsection for details). Further, the value of the parts must add up to the value of the whole such that **S** qualifies as a Σ-algebra. A measure on **S** is axiomatically introduced and set equal to 1 such that the associated value scale *v* will show ratio-like[4] properties. This yields an equation in the representational structure, namely the tensor representation of the accounting equation **(x||x)(y||y)** = 1.

Moreover, by construction, the measures assigned to strategies in **S** are additive. A strategy being defined within a coarser partition may be conveyed, alternatively, as the superposition of strategies defined within a finer partition.

Taxonomic Partitions:

In accounting, the structure of the balance sheet, which herein includes the income statement, is based upon partitions. A typical example is shown in Figure 3. This structure qualifies as a taxonomic partition, as this has been axiomatically framed by Kay (1971).

The features characterising the balance sheet as a taxonomic structure are as follows:

(i) there is a hierarchy; (ii) the hierarchy is derived from partitions; (iii) partitioning yields a *partial* order; (iv) the logic underlying the partitioning procedure is time-oriented; (v) as a result of arising out of partitions, assets T-accounts are not matched to liabilities T-accounts on an individual basis, but only at the aggregated level.

[4] *v* is bounded, the maximum value being 1. The maximum is arbitrary by means of re-scaling, however, the function *v* embodies a 'curvature' characteristic such that one may multiply it by fractions of the maximum, as with any true ratio scale, but one cannot use multiples thereof.

Assets	**1-000**	**Income**	**4-000**
Current Assets	1-100	Services	4-101
Bank	*1-101*	Materials	4-102
Petty Cash	*1-102*	Bank Interests	4-103
Accounts Receivable	*1-103*		
Fixed Assets	1-200		
Office Equipment	*1-201*	**Cost of Goods Sold**	**5-000**
Motor Vehicle	*1-202*		
Liabilities	**2-000**	**Expenses**	**6-000**
Current Liabilities	2-100	Advertising	6-100
Bank Overdraft	*2-101*	Bank Fees	6-200
Credit Card	*2-102*	Telephone	6-300
Accounts Payable	*2-103*	Subscriptions	6-400
Long Term Liabilities	2-200		
Bank Loan	*2-201*		
Motor Vehicle Loan	*2-202*		
Equity	**3-000**		
Capital	3-100		
Drawings	3-200		

Figure 3: The taxonomic structure of the balance sheet

The resulting structure is mapped onto a set of numbers having a quotient structure such that 1-101 may be interpreted as 1-100 divided by 10 or as 1-000 divided by 100.

Kay's (1971) axioms yield a finite set with terminal T-accounts (e.g., 1-202 in Figure 3) which strictly include no other and such that they collectively span the universal set **S**. In using vectors and matrices to represent accounting numbers, reliance is place on terminal T-accounts because they provide a basis for the applicable vector space (e.g., Cruz Rambaud et al., 2011).

I am suggesting that economic valuation be framed by reference to the structure of **S**. I am requiring the observable, physical quantities (i.e., the inputs **(x|** and outputs **(y|**) to be inserted, as known parameters, into the accounting equation (i.e., **(x||x)(y||y)** = 1) such that a constraint is axiomatically imposed on the measure associated with the strategy. With such constraint, a (non-deterministic) algorithm can then be worked out that yields consistent costs, the economic values **|x)**, based on a subjective valuation of the desired outputs, the **|y)**.

Derived Measurement and 'Lifting' the Formal Language:

Given the foregoing, in particular that the input-output pair—not the commodity—is the object of value measurement, assigning values to resources qualifies as a derived measurement procedure. Costs, prices, values are determined only if a representational structure is in place. What follows is inspired by Ellerman's (2013) 'lifting' discussion. The idea is to recognise a structure that accounts for physical quantities and is denoted **X**. However, the key structure for measurement purposes is one that accounts for the relational system of ratios wherein inputs are matched to outputs.

This latter structure refers to strategies (i.e., purposeful processes that change resources over time) and is denoted **S**. The two structures are intertwined by means of two mappings, ǭ from **X** to **S**, and the value scale *v* from **S** to $\mathbb{Q}$ (i.e., the set of rational numbers). It is then imposed that ǭ lifts the scale *f*, the *derived* measurement that maps **X** to ℚ, by means of which values, costs and prices, are assigned.

By lifting, it is meant that *v*[ǭ(*x*)] = *f*(*x*), for any *x* in **X**. Figure 4 illustrates the lift concept: ǭ is introduced as the morphism[5] that maps resources in **X** to ratios in **S**; further, with the condition that *f*(**X**) = 1, the value *v* becomes a probability measure mapping ratios to ℚ.

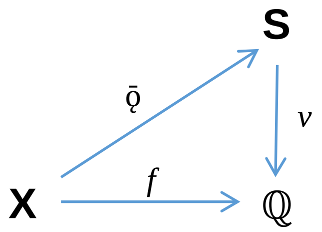

Figure 4: ǭ is the morphism that lifts the scale *f*

Requiring that ǭ lift the scale *f* not only establishes *f* as a derived measurement in terms of **S**, it takes the framework beyond basic extensiveness.

At this point value measurement becomes unavoidably associated with accounting measurement. The term commodity no longer applies and is replaced by 'resource subsets' such as 'in-process inventory' or 'plant equipment'. Accounting measurement first classify the resource subsets by reference to some underlying process. Only after classification has been completed, valuation may take place. ǭ expresses the nominal part of the scale *f*. Classification being completed, probabilities are assigned to the inputs-outputs pairs and the scale *v* obtains.

Next, I justify the need for all the above trouble. The neoclassical framework to analysing decisions is not just prone to empirical refutation; there is a purely theoretical issue as well.

6th idea: The "inescapable accountingness of economic questions" (Klamer & McCloskey, 1992: Accounting as the master metaphor of economics).

When the demarcation that segregates economics from accountancy was not as clear as today, theoretical economists had a practical understanding of accounting procedures with which they framed their research questions. Commenting on Nobel prize-winner John Hicks' (1942) *The Social Framework: An Introduction to Economics*, Klamer and McCloskey (1992) write:

> *The book instructs the reader to distinguish stocks and flows, and to recognize how economic magnitudes are codetermined in a system of accounts. The student learns to think about economic events in the first instance as altering the accounts. In other words, the economic student is to begin his intellectual journey equipped with accounting tools. For a few years Hicks's book was popular, and accounting and economics walked together. (Klamer & McCloskey, 1992: p.151)*

Their point is that Samuelson's work decisively banned the accounting programme from economic theory. Samuelson established the new, hegemonic paradigm of economic theory

[5] ǭ is a correspondence, not a function; it maps inputs to outputs by reference to the several ratios.

by raising the formal standards of research in this area. However, his approach did not leave room for accounting principles and procedures. This is a problem in view of the "inescapable accountingness of economic questions" (idem: p.152). Failure to accommodate accounting implies that Samuelson's paradigm is unable to deal with the full range of economic issues.

Cost as opportunity cost:

Another reference in support of the decision-making approach being championed here is Nobel-prize laureate James Buchanan, who in his book *Cost and Choice* (Buchanan, 1969) refers to the following Adam Smith's famous example to make a relevant point.

> *If among a nation of hunters ... it usually costs twice the labour to kill a beaver which it costs to kill a deer, one beaver should naturally exchange for or be worth two deer. The classical theory of exchange value is summarized in this statement... Normal or natural value in exchange is determined by the relative costs of production. (Buchanan 1969: p.12)*

Buchanan explains that this tale holds true only in the very simple economic system consisting of a single, homogeneous input. Therein, cost means opportunity cost and as such is a concept readily applicable to decision-making. When more complex models are introduced, for example one requiring a multi-dimensional economy, the concept of cost becomes problematic; without defining cost, how may rational decision-making be properly dealt with? For example, Buchanan comments on models that introduce money as a numeraire:

> *If costs are $10, the producer must expect a value of at least $10. The postulate of rational behavior along with the presumption that the numeraire is positively desired still implies that expected value be equal to or above costs. But what now determines costs? No longer is the theory simple enough to concentrate our attention on one moment of decision, one act of choice. Instead of this, we now must think of a chain of interlinked decisions over varying quantities of output, over separate time periods, and over many decisionmakers.(Buchanan 1969: p.14)*

Aggregation:

Further, if a set of several, inhomogeneous resources is required for completing a single production process, an issue must be addressed, namely aggregation. Aggregation is an intractable issue within economic theory (e.g., the Cambridges' controversy[6]). I claim this is due to economists' conceiving the production function independently of any concurrent discussion about aggregation and the measurement of value.

The issues:

The theoretical issues are as follows: The "inescapable accountingness of economics" means that one needs a dynamic formalism wherein value is inherently uncertain.

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[6] Cambridge in the USA vs. Cambridge University in the UK. The following quote illustrates the issue: "… the production function has been a powerful instrument of miseducation. The student of economic theory is taught to write Q = *f*(L, K) where L is a quantity of labor, K a quantity of capital and Q a rate of output of commodities. He is instructed to assume all workers alike, and to measure L in man-hours of labor; he is told something about the index-number problem in choosing a unit of output; and then he is hurried on to the next question, in the hope that he will forget to ask in what units K is measured. Before he ever does ask, he has become a professor, and so sloppy habits of thought are handed on from one generation to the next." (Robinson, 1953)

Indeed, as opposed to the neoclassical framework wherein pure profits are inexistent (by construction, given the mathematical characteristics of the formalism), in accounting every new transaction has the potential to update the value of the firm by means of income recognition.

Production processes are typically many-to-many (Willett, 1988); there are many inputs to many outputs. This is not how the production function is conceived by economists who usually assume a many-to-one approach: inputs yield a single output. Thus, they are able to define, for example, marginal cost, a concept that cannot be applied to many-to-many correspondences[7]. Herein, the output quantity $\mathbf{q_j}$ represents many physically distinct outcomes to a production process encoded by **T**. Thus, for example, the standard problem of joint product costing is inherently embedded in the framework.

The understanding that underlies the concept of cost is opportunity cost. Given two alternative strategies, a measure is associated with one strategy by reference to the other strategy. Value is purely relational.

Economic resources are not defined by their intrinsic physical characteristics. With the view of valuing an apple pie, it would therefore be incorrect to look for the market prices of each of the ingredients, prices being set by consumers and producers. Instead, the inhomogeneous set of ingredients (apples, sugar, flour, etc.) are considered as one input to baking the pie such that their individual value is relevant, but not the single determinant of costs. Indeed, without flour, all the other ingredients are worthless in terms of the pie.

The proposed solution:

Conceive 'strategy' further to a concurrent analysis of accounting processes which include production, opportunity cost, aggregation, and the measurement of value. My ideas follow:

(i) by construction, total value remains constant within a strategy;

(ii) the set of resources is partitioned into time-indexed inputs and outputs and the values assigned to resources must add up to 1 at all times;

(iii) the mathematical object representing the strategy is the bi-linear operator $|\mathbf{x})(\mathbf{y}|$ that maps quantity vectors in the input space to quantity vectors in the output space;

(iv) to go beyond just the physical dimension of strategies, the accounting equation, $(\mathbf{x}|\mathbf{x})(\mathbf{y}|\mathbf{y}) = 1$, is introduced as a value consistency requirement that embodies the teleological nature of strategies, that is, that costs must be consistent with the physical ratios associated with the strategy;

(v) the operator $|\mathbf{x})(\mathbf{y}|$ is defined in the tensor space $(\mathbf{P}\times\mathbf{Q})$, with an orthogonal (but not orthonormal) basis given by the pairs (q_i, p^j)[8]; thus, it becomes possible to represent the partitioning of the universal set of resources as a direct sum decomposition in the representational, tensor space;

(vi) upon direct sum decomposition, the input-output ratios matrix $|\mathbf{x})(\mathbf{y}|$ is given a block diagonal representation that satisfies $(\mathbf{q_i}|\mathbf{I}|\mathbf{p^j}) = (\mathbf{1}|\mathbf{QP}|\mathbf{1}) = 1$ whereas **I** is the identity matrix and **QP** is the product of two diagonal matrices, **Q** accounting for quantities and **P** for values;

[7] In economics, a correspondence is the generalization of a function and may be referred also as a multivalued function. Given the sets A and B it is a map f:A→P(B) from A to the power set of B.

[8] Lower and upper indices are introduced herein as a notational convention such that the tensor space—which qualifies as a vector space as well—is associated with a (i+j)-dimensional basis of ordered pairs (q_i, p^j). Further, this notation is consistent with Einstein's summation convention.

(vii) the measure 1 informs the constant value of the whole bundle of resources such that by partitioning the bundle the individual values of each resource subset obtain; they are calculated by projecting the operator $\mathbf{V} \overset{\text{def}}{=} \mathbf{QP}$ onto a basis of its space. The quantity vector space and the valuation one are now space projections of the general tensor space $\mathbf{V} = |\mathbf{P}; \mathbf{Q}|$. As such, **Q** is the diagonal matrix that projects value onto the present vector-basis of **V** while **P** is the diagonal matrix that projects value onto the future vector-basis of **V**;

(viii) the accounting equation, $(\mathbf{x}|\mathbf{x})(\mathbf{y}|\mathbf{y}) = 1$, holds only when a basis is known (a partition has taken place); this is interpreted as a solution concept rather than a solution (see next section for details);

In short, the proposed solution is based on the formalism. Such understanding is captured and illustrated by this other quote from Hicks:

> *I have actually seen business decisions being made on the basis of projected balance sheets. I think that is the rational way to make a business decision. A lot of these mathematical models, including some of my own, are really terribly much in the air. They lost their feet off the ground. (Klamer & McCloskey, 1992: p.152)*

I am suggesting the concept of a production process to encompass and extend the economists' concept of a production function by using tensors. They address physical transformations, aggregation, and the measurement of value simultaneously.

Tensors are linear transformations. Production functions are typically non-linear[9]. However, linearity does not restrict the formalism's ability to deal with any set of data. First, because accounting measurement takes place 'atop' of physical measurements. Decision-makers are assumed to know what resources are required for a desired output. They then use this knowledge to identify the quantities deemed relevant at each instant for accounting analyses.

Next, because whatever the relevant quantity vectors at some earlier time, $\mathbf{q_i}$, and at some later time, $\mathbf{q_j}$, a linear transformation **T** can always be constructed to represent the physical productive transformation taking place between the two moments, $\mathbf{q_j} = \mathbf{Tq_i}$. The only exception is when the initial quantities are all zero—a case that does not limit the formalism since it can be ruled out that an output is obtained without any input. Solvability is assured since equation $\mathbf{q_j} = \mathbf{Tq_i}$ implies m^2 unknowns (i.e., the number of elements in **T** for m-dimensional **Q**-spaces) being restricted by m equations. The $m(m\text{-}1)$ degrees of freedom are sufficient for solving the equation, except when $\mathbf{q_t} = \mathbf{0}$. Notice the solution **T** is not unique.

[9] For example, the Cobb-Douglas function $Q = aL^{\alpha}K^{\beta}$. This is a function with appealing mathematical properties: it is easy to differentiate such that finding the marginal costs of labour and capital is straightforward. However, the problem with it is that it has no grounds in physical reality—it lacks micro foundations—nor (quite surprisingly) does it inform anything about aggregate production, despite a long history of econometric data (Felipe & McCombie, 2005). Zerner (2016) proves that the Cobb-Douglas function is a necessary, purely mathematical, consequence of an accounting identity.

7th idea: *Refer to Game Theory and re-conceptualise the accounting equation as the solution concept applicable to valid strategies. The concept of strategy replaces the concept of production function for the purposes of economic decision.*

The discussion to follow makes reference to Game Theory. Consider that production may be framed as a game consisting of two players, nature and the decision-maker.

In production, nature is a player without a strategic interest in the outcome of the game and as such its inclusion aims to incorporate randomness into the formalism. This randomness is, however, restricted by the laws of physics and by engineering constraints such that the associated probability distribution accounts for the variations that take place in production.

The solution concept:

Trading or production processes, whatever the case, the point is that the solution concept be such that it supports modelling the decision-maker's ability to learn. Value numbers that are assigned to resources should provide decision-makers with confirmation that their beliefs about production and other agents' behaviour hold true. If so, the solution concept satisfies the criteria of self-confirming equilibria (Fudenberg & Levine, 1993), namely that players

(i) choose their strategies with the view of reaching their pre-defined goals,
(ii) based on their beliefs about what the other players (including nature) will choose,
(iii) and based on the prohibition that their beliefs contradict any available empirical evidence, including any evidence gained as the chosen process unfolds.

This implies consistency between beliefs and any available empirical data that arises out of observing the process of interest. This is the accounting approach to measurement whereby income is up-dated from time to time to reflect the knowledge gained from the entities' own behaviour and its interaction with the economic environment. Within a setting of uncertainty, as is typical, measurement is thus associated with a dynamic process of valuation.

Inherent uncertainty:

Further, value measurement becomes inherently uncertain. For a naïve account of what this entails, imagine an older brother and a younger sister. A doll is present in the room where they are playing. Playing with the doll yields no satisfaction to the boy but he believes his sister's playing with it provides her with pleasure. He takes the doll with the sole purpose of preventing her from playing. Should she complain, he obtains evidence the doll is valuable to her. The satisfaction he derives from preventing the girl from playing determines the value that he assigns to the doll. If she does not complain, the doll is worthless to him.

When choosing among his own toys, the boy considers the relative pleasure that they provide him. This is the logic of traditional economic valuations. When choosing his sister's toys, he considers her displeasure as a measure of his own satisfaction. This requires a game theoretical approach to measurement whereby values are inherently uncertain, and remain so, until an observation reveals the outcome of a strategy.

The nature of the attribute associated with value:

Thus, the 'stuff' that is measured when values are assigned to resources is how consistent the decision-maker is. If he or she knows how a process unfolds, the assigned values must be such that the input-output ratios are consistent therewith. Values are predicated on the available information about the underlying input-output relationships. The performance measure indicated in Figure 1 is set equal to one to account for that consistency (i.e., 1 expresses a 100% consistency). When observed input-output ratios are not consistent, then the accounting equation requires the decision-maker to up-date the original values such that the aggregate value of all resources associated with the underlying strategy returns to one.

It is interesting to note that in accounting theory, Christensen & Demski (2002) have been criticising the understanding that accounting numbers are representations of economic value. Consequently they have, among others, introduced the information content perspective of accounting. Although different, the approach herein tries to reconcile the foregoing mathematical interpretation of value (i.e., a scale) with their information content perspective. The level at which the resource set is partitioned, coarser or thinner, provides information about the underlying process wherein the resources are actively used.

Further, from a mathematical perspective, interpreting knowledge as the attribute of value is consistent with the views of Krantz et al. (1971: 123-124) and Roberts (1984: 129) about fundamental measurement. To them, subjective probabilities qualify as fundamental extensive measurements within the social sciences. Accounting numbers are framed as the fundamental measurement of a decision-maker's subjective perception of economic processes.

8th idea: Final remarks

The multiplication $|\mathbf{x}^{\mathbf{i}})(\mathbf{y}_{\mathbf{j}}|$ is the outer product of $|\mathbf{x}^{\mathbf{i}})$ by $(\mathbf{y}_{\mathbf{j}}|$. This outer product is the multiplicative representation of a T-account. By focusing on the vectors $(\mathbf{x}|$, $|\mathbf{x})$, $(\mathbf{y}|$, and $|\mathbf{y})$, the consistency solution concept requires the total value of resources to be constant and equal to one: $(\mathbf{x}|\mathbf{x}) = 1$ earlier; and $(\mathbf{y}|\mathbf{y}) = 1$ later. This provides a stock-oriented (i.e., commodity) interpretation of value. Focusing on $|\mathbf{x})(\mathbf{y}|$ provides the flow-oriented interpretation such that it encodes a multiplicative version of the accounting equation.

A typical process consists of several transformations, each given by an identity[10] T-account. A process taking place in three steps is encoded by three T-accounts, say $|\mathbf{x})(\mathbf{w}|$, $|\mathbf{w})(\mathbf{z}|$, and $|\mathbf{z})(\mathbf{y}|$. If so, $(\mathbf{x}||\mathbf{x})(\mathbf{w}||\mathbf{w})(\mathbf{z}||\mathbf{z})(\mathbf{y}||\mathbf{y}) = 1$ is the applicable accounting equation. Associativity yields $(\mathbf{x}||\mathbf{x})(\mathbf{y}||\mathbf{y}) = 1$.

The multiplicative T-account is interpreted as a projection operator that transforms costs and quantities during the process' timeframe. Given earlier quantities $(\mathbf{x}|$, the T-account $|\mathbf{x})(\mathbf{y}|$ transforms $(\mathbf{x}|$ into the later quantities $(\mathbf{y}|$: $(\mathbf{x}|\mathbf{x})(\mathbf{y}| = (\mathbf{y}|$. Given the later valuation $|\mathbf{y})$, the T-account transforms its associated cost vector into the earlier one $|\mathbf{x})$: $|\mathbf{x})(\mathbf{y}|\mathbf{y}) = |\mathbf{x})$.

Two concurrent interpretations apply to the accounting equation. It encodes a causal relationship whereby earlier quantities are transformed into the later ones. As such, $|\mathbf{x})(\mathbf{y}|$ is the representation of economists' production function. Second, the multiplicative structure also encodes a teleological relationship defined by the process' purposes.

The process timeframe is partitioned into several moments when observations take place. Each moment is assigned an index; for example with two moments, i is assigned to the initial and j to the final moment. The resource set is partitioned at each such moment in view of

[10] See Appendix 1 for a discussion about the term 'identity'.

what is observed. Each of the resulting resource subsets is assigned two numbers, q_i and p_i, i = 1,..,*m*, where *m* counts the number of resource classes (qualities); q_i expresses the extensive measurement of the i^{th} resource (quantities); and p_i is the value associated with the i^{th} resource (dimensional valuation). Their product gives the (a-dimensional) value of the resource subset.

Conclusion

The accounting equation is a tautological truth. Whatever the agent, which in particular may be the whole economic system (i.e., a coalition of people and companies), the value of their assets equals the value of their liabilities.

This, however, does not provide any useful information, not at the aggregated level. For valuation to become meaningful, the resource set is partitioned by reference to several strategic processes, the level of information which this entails depending on the coarseness or finesse of said partitions.

Resources are assigned values which inform how they pertain to a dated net of input-output relationships. Further, partitioning also determines the different agents controlling those resources. The key for the proper interpretation is that values assigned to resources must be consistent with the beliefs associated with the agents controlling said resources.

The mathematical framework is constructed such that its basic element is the T-account. It expresses both a step in a process and the existence of a stock of inputs at an earlier time and a stock of output at a later time. Being constructed as a combination of T-accounts, strategies are teleological processes connecting the inputs to outputs.

For lack of space, the issue of income cannot be discussed. As mentioned in the introduction, the framework is conservative: value and costs remain constant when an event takes place and the corresponding transformation is formally recorded. This seems to suggest that income is also constant; however, a re-conceptualisation of income is possible such that income may vary even within a conservative framework.

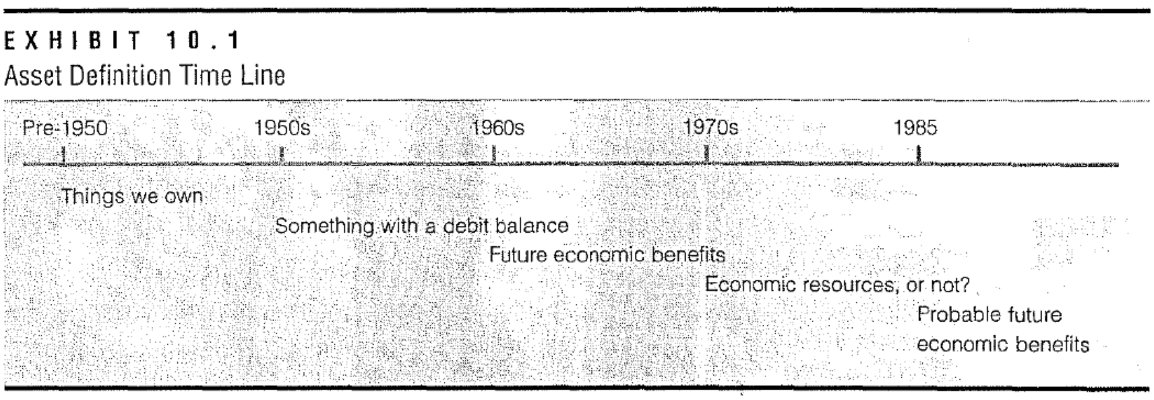
APPENDICES

1. The algebraic approach to defining mathematical objects

As mentioned before, economic theorists nowadays seem unaware of accounting theory[11]. As I see it, this is mostly due to a failure by accounting theorists to provide scholars outside their field with a sound theory of accounting, DEB in particular. Indeed, at some point in history (in the 1970s), theoretical accountants gave up on developing a genuine accounting approach to DEB and started to rely on economic theory.

The present discussion is foreign not just to economists but to most accountants as well. Mainstream accountants have come to accept mainstream economics, particularly the formal language implied by general equilibrium. General equilibrium is at the basis of fair value and the current conceptual framework which underlies most accounting standards (both FASB's and IASB's)[12]. Accordingly, for example, the basic concept of 'asset' is defined by reference to economics. The intuition behind an asset is that of a commodity that commands a cash-flow which can be measured and attributed a present value. This, however, has not always hold true.

Evans (2003, ch.10) summarises the shift in defining the concept of 'asset' over time by means of his Exhibit 10.1, reproduced below. My point is that before the economic formal framework took over accounting theory, accountants tried diverse definitions, one of which provides a historical antecedent for the approach being championed in this paper.



> *Something represented by a debit balance that is or would be properly carried forward upon a closing of books of account. … on the basis that it represents*

[11] 'In fact, most are startled to learn of the existence of academic articles on accounting. Academic accounting? One might as well have academic plumbing.' (Klamer & McCloskey, 1992). For a complementary view on this, see Shubik (2011).

[12] FASB stands for the Financial Accounting Standard Board, in America; IASB stands for the International Accounting Standard Board, in Europe and most of the world.

either a property right or value acquired or expenditure made which … is applicable to the future.[13]*(p.217)*

Unclear as this may be to laypeople, unfamiliar with the accounting parlance, I claim this provides the first attempt at defining the concept of asset in genuine accounting grounds.

In that vein, the subsequent attempt, much less known even among accounting theorists, is due to Ellerman (1985; 2014a). He proposes two defining statements: (i) T-accounts are objects satisfying the axioms of algebraic groups and such that (ii) they encode the accounting equation into the group's neutral element, the zero-term. Due to space and focus constraints, I refrain from elaborating. However, I take the opportunity to clarify what I consider is the algebraic approach to defining mathematical objects.

When first learning about numbers, children are not exposed to complex numbers. They start with natural numbers. This is because a natural number can be readily associated with the quantity in a countable set such that children are able to grasp, *concretely and empirically*, what a number is supposed to mean. Further, addition is explained by the union of countable sets and multiplication by sequential additions.

At some stage, however, the visual connection involving addition and multiplication with the algebra of sets no longer applies and children are expected to develop an abstract reasoning. Equations are introduced, say $2x = 1$, such that numbers can be conceptualized by reference to equations, which are mental constructs.

With the foregoing in mind, it is possible to understand how the single rational number ½ is conceptualized to be the result of an ordered pair of two natural numbers that are the solutions to the equation $2x = 1$. This equation admits infinite solutions (e.g., 1/2, 2/4 or 32/64). The human mind, however, is not comfortable with objects that cannot be determined precisely. Thus, mathematicians rely on the quotient concept to deal with the issue. A rational number is defined as *a set* of ordered pairs of natural numbers that satisfy the following rule: x/y is equivalent to x'/y' if and only if the equation $xy' = x'y$ holds true. Each rational number is therefore an equivalence class of ordered pairs of natural numbers that satisfy a given equation.

The foregoing discussion is expected to help make sense of Ellerman's (1985; 2014a) definition of T-accounts and of the balance sheet being constructed as a T-account. T-accounts are equivalence classes of input-output relationships existing in an underlying relational system of accounting transformations. The idea of an equation, the accounting equation, whose solutions define the valid T-accounts, provides the algebraic basis for a formal definition of accounting terms without the need to refer to external (i.e., semantic) terms arising from economics or elsewhere.

The accounting equation is a representational statement that expresses a strategy wherein earlier resources are matched to later (desired) outcomes. Assets are thus the accounting representation of those earlier resources while liabilities represent the later outcomes. When a monetary unit of measurement is introduced, that view about liabilities can be made consistent with the usual understanding of what the term liability entails.

[13] *Accounting Terminology Bulletin No.1, in* AICPA, 1953, par.26.

2. Basic Concepts of Linear Algebra

The possibility of relying on Linear Algebra to address accounting issues has been recognised by some theorists long ago (e.g., Mattessich, 1957).

A *vector space* is an algebraic structure that consists of two sets such that their elements satisfy a certain list of axioms. The first set V contains vectors and the second set F is called a field. Herein the field is $\mathbb{Q}$, the rational numbers, unless otherwise indicated. Any introductory book on linear algebra discusses the axioms that define a vector space (e.g., Hoffman & Kunze, 1971: ch.2).

Given vector spaces V and W, a *linear transformation* is a function $\mathbf{T}$: V $\mapsto$ W such that for all $\mathbf{v}_1, \mathbf{v}_2 \in$ V and all $a, b \in \mathbb{Q}$ it is true that $\mathbf{T}(a\mathbf{v}_1 + b\mathbf{v}_2) = a\mathbf{T}(\mathbf{v}_1) + b\mathbf{T}(\mathbf{v}_2)$[14]. When V = W the function $\mathbf{T}$ is sometimes called an *operator* (i.e., an operator is an endomorphism preserving vector addition and scalar multiplication).

Any linear transformation between finite vector spaces can be represented by a conveniently chosen *matrix*. When approaching a problem in linear algebra it may be useful to switch back and forth between the more abstract linear transformation approach and the more concrete (i.e., visual) matrix approach.

The symbol $\mathbf{T}_{mxm}$ expresses both an operator $\mathbf{T}$: $\mathbb{Q}^m \mapsto \mathbb{Q}^m$ and its associated $m\times m$ matrix [**T**]. Further, composing two operators $\mathbf{T}^1$ and $\mathbf{T}^2$ (i.e., $\mathbf{T}^2\circ\mathbf{T}^1$) and multiplying their respective matrices (i.e., $\mathbf{T}^2\cdot\mathbf{T}^1$) are accounted for by juxtaposing the two symbols (i.e., $\mathbf{T}^2\mathbf{T}^1$) with the interpretation that $\mathbf{T}^1$ is performed first and $\mathbf{T}^2$ next.

Given an $m\times m$ square matrix **T**, we can choose a horizontal partitioning of its rows and a vertical partitioning of its columns. When the same partition is used for both rows and columns we refer **T** as a *blocked matrix* such that $\mathbf{T}_{ij}$ is its (i, j)th block.

Blocked matrices are important herein because I deal with multiplication block-wise. Say a 3×3 **A** matrix is blocked under the structure (2, 1), that is, the set of rows (resp. columns) is partitioned in two subsets, one with two rows (resp. columns) and the other with one. When **A** is multiplied by another 3×3 **B** matrix, which also has been blocked (2, 1), the resulting blocks of AB_{ij} obtain by means of $\Sigma_k\, a_{ik}b_{kj}$, k = 1, 2.

In matrix notation the foregoing reads as follows,

$$\mathbf{A} = \left(\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ & A_{11} & A_{12} \\ a_{21} & a_{22} & a_{23} \\ \hline & A_{21} & A_{22} \\ a_{31} & a_{32} & a_{33} \end{array}\right) \quad \mathbf{B} = \left(\begin{array}{cc|c} b_{11} & b_{12} & b_{13} \\ & B_{11} & B_{12} \\ b_{21} & b_{22} & b_{23} \\ \hline & B_{21} & B_{22} \\ b_{31} & b_{32} & b_{33} \end{array}\right) \quad \mathbf{AB} = \left(\begin{array}{c|c} \Sigma A_{1k}B_{k1} & \Sigma A_{1k}B_{k2} \\ \hline \Sigma A_{2k}B_{k1} & \Sigma A_{1k}B_{k2} \end{array}\right)$$

(2, 1) (2, 1) (2, 1)

[14] I adopt the notational convention such that vectors are denoted by bold lower case letters while linear transformations are denoted by bold upper case letters.

The simplest blocked matrices are the *block diagonal* ones such that all blocks off diagonal contain zeroes. In such cases we write $\mathbf{T} = \text{diag}(\mathbf{T}^1, \mathbf{T}^2, ..., \mathbf{T}^r)$ and **T** is said to be *the direct sum* of $\mathbf{T}^1, \mathbf{T}^2, ..., \mathbf{T}^r$ (i.e., $\text{diag}(\mathbf{T}^1, \mathbf{T}^2, ..., \mathbf{T}^r) \stackrel{\text{def}}{=} \mathbf{T}^1 \oplus \mathbf{T}^2 \oplus ... \oplus \mathbf{T}^r$).

The following theorem[15] deals with an operator's direct-sum decomposition.

> ***T****: V ↦ V is an operator, $V = U_1 \oplus U_2 \oplus ... \oplus U_k$ is a direct sum decomposition of V into* ***T****-invariant subspaces $U_1, U_2, ... U_k$. If $\boldsymbol{B}_i$ is a basis for U_i, then let the set $\boldsymbol{B} = \boldsymbol{B}_1 \cup \boldsymbol{B}_2 \cup ... \cup \boldsymbol{B}_k$. Relative to the basis* ***B*** *for V, the matrix of* ***T*** *is the block diagonal matrix $\text{diag}(\boldsymbol{T}^1, \boldsymbol{T}^2, ..., \boldsymbol{T}^k)$ where $\boldsymbol{T}^i$ is the matrix relative to $\boldsymbol{B}_i$ of the restriction of* ***T*** *on $U_{i.}$ (O'Meara, Clark, & Vinshonhaler, 2011: p.21).*

A **T**-invariant subspace U_i of V is such that for a vector $\mathbf{u_i} \in U_i$ one has $\mathbf{T}(\mathbf{u_i}) \in U_i$. This theorem states that there exists a convenient basis for V such that the operator **T** is associated with the block diagonal matrix $\mathbf{T} = \text{diag}(\mathbf{T}^1, \mathbf{T}^2, ..., \mathbf{T}^k)$, whereas each $\mathbf{T}^i$ is defined over the respective subspace U_i.

This theorem provides the key mathematical result for the process-based approach to accounting measurement being championed in this working paper.

An accounting transformation—trading or production—will be accounted for by an operator **T**. This operator acts on a subset of the resource set (i.e., on those resources being traded or transformed by production). This operator does not affect the complement subset of the resource set (i.e., the other resources not being traded or transformed by production). **T** represents the observable accounting event of interest, typically a productive transformation or a trading transaction.

Further, the resource set is assumed to be 'indefinite'[16] before a basis for the vector space is chosen and **T** may be rendered by a matrix. The direct-sum decomposition of the vector space induced by the operator **T**, as per the theorem above, yields subsets of the resource set that are process-based. Resources are thus identified in view of a given process rather than in view of their physical characteristics.

I am dealing with tensors and tensor spaces that result from pairing vectors with covectors. Given the vector space V, the *algebraic dual space* V* is defined as the set of all linear maps $\mathbf{u}: V \mapsto \mathbb{Q}$. V* is itself a vector space when the following apply:

(i) $(\mathbf{u_1} + \mathbf{u_2})(\mathbf{v}) = \mathbf{u_1}(\mathbf{v}) + \mathbf{u_2}(\mathbf{v})$

(ii) $(a\mathbf{u_1})(\mathbf{v}) = a(\mathbf{u_1}(\mathbf{v}))$

for all $\mathbf{u_1}$ and $\mathbf{u_1} \in V^*$, $\mathbf{v} \in V$, and $a \in \mathbb{Q}$.

Elements in the algebraic dual space V* are called covectors. Pairing a covector **u** in V* with a vector **v** in V defines the bilinear mapping $(\mathbf{u}, \mathbf{v}): V^* \times V \mapsto \mathbb{Q}$. In finite-dimensional cases, V* has the same dimension as V.

[15] This theorem is a generalisation of the spectral decomposition theorem.

[16] See Ellerman (2013). For some intuition, consider a commodity bundle such that some goods, but not all, are sold in a single transaction. Thus, the bundle has been partitioned in two classes: the goods that have been sold and those that have not. A distinction has arisen out of trading, not in view of any possible physical characteristics pertaining to the sold goods, as opposed to the unsold ones.

The next section introduces the vector quantity space, the Q-space, and its dual convector P-space, both being isomorphic to $\mathbb{Q}^m$.

2.1 The quantity vector space Q and the valuation covector space P

The *quantity vector space*, Q-space, is introduced and is posited isomorphic to $\mathbb{Q}^m$. I denote a q-vector by a lower case bold letter within left semi-brackets. The symbol $(\mathbf{q}|$ denotes a quantity vector; a set of numbers $\{q_i, i = 1, 2, \dots, m\}$ exists such that $(\mathbf{q}| = \Sigma_m q_i\mathbf{e_i}$, whereas $\mathbf{e_i}$ is an element in the canonical basis of $\mathbb{Q}^m$. This notation identifies quantity vectors $(\mathbf{q}|$ with $1{\times}m$ matrices, horizontal lists of numbers; thus,

$$(\mathbf{q}| = (q_1, q_2, \dots, q_m|_{1\times m}$$

Let the symbol $(\mathbf{1}|$ denote the particular quantity vector for which all entries are equal to 1[17]. In matrix notation,

$$(\mathbf{1}| = (1, 1, \dots, 1|_{1\times m}$$

Left-multiplying an $m{\times}m$ matrix $\mathbf{D}$ by $(\mathbf{1}|$ yields the unique $1{\times}m$ matrix $(\mathbf{1}|\mathbf{D}$. In particular, if $\mathbf{D}$ is diagonal, the m entries in $(\mathbf{1}|\mathbf{D}$ are the diagonal elements in $\mathbf{D}$. Conversely, given any $1{\times}m$ matrix, a unique diagonal $m{\times}m$ matrix $\mathbf{D}$ exist such that $(\mathbf{1}|\mathbf{D}$ is this matrix—suffice to assign the elements in the $1{\times}m$ matrix to the diagonal entries in $\mathbf{D}$ and zero otherwise. There is a bijection between the set of $1{\times}m$ matrices and the set of diagonal *mxm* matrices.

The motivation for introducing $(\mathbf{1}|$ and for mentioning the foregoing bijection between vectors and diagonal matrices is this: resource subsets, commodities, and transformations of resource subsets, production or trade, are both represented in the formal language by means of a single mathematical object, the *mxm* matrix.

Consider the sequence of time intervals (t-1, t], for t = 1, …, f. Assume an initial transformation takes place within the first interval (0, 1] such that its representation is an $m{\times}m$ diagonal matrix $\mathbf{T_{0,1}}$: Q → Q. Call the vector that results from multiplying $(\mathbf{1}|$ by $\mathbf{T_{0,1}}$ the *initial quantity* $(\mathbf{q_1}|$. If so, $(\mathbf{q_1}| = (\mathbf{1}|\mathbf{T_{0,1}}$.

For example, assume a Q-space with two dimensions which are interpreted as two resources, say apples and oranges. If the initial quantities at time 1 are two apples and three oranges, there is a unique diagonal $\mathbf{T_{0,1}}$ that yields the proper initial quantity vector,

$$(\mathbf{1}|\mathbf{T_{0,1}} = (\mathbf{1, 1}| \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = (\mathbf{2, 3}| = (\mathbf{q_1}|$$

Whatever the quantity $(\mathbf{q_t}|$, it is possible to express the operator $\mathbf{T_{0,t}}$ as a diagonal matrix which multiplied by $(\mathbf{1}|$ yields the given vector, $(\mathbf{q_t}| = (\mathbf{1}|\mathbf{T_{0,t}}$. Referring to the index t = 1,…,f, a quantity at time t can be accounted for by a diagonal matrix $\mathbf{T_{0,t}}$.

A key structural feature embodied within the foregoing notation is this: whatever the quantities at some earlier time t and whatever the quantities at some later time t’ a matrix $\mathbf{T_{t,t'}}$

[17] Ijiri (1965: p.88) calls this vector a “totalizer”: in the context of multi-dimensional accounting, he uses it to calculate the single, balance-sheet value that aggregates the values of a set of accounts. Tippett (1978) also uses it, except that his analysis is restricted to three dimensions only.

can always be constructed to represent the perceived changes in quantities between the two instants. The only exception is when the initial quantities are all zero, a case that does not limit the formalism since we may rule out the possibility that an output quantity could be obtained without any input. Formally,

Solvability theorem – $\forall\ (\mathbf{q_t}| \neq 0,\ \forall\ (\mathbf{q_{t'}}|,\ t < t',\ \exists\ \mathbf{T_{t,t'}} : (\mathbf{q_{t'}}| = (\mathbf{q_t}|\mathbf{T_{t,t'}}$

This is true because the equation implies m^2 unknowns (i.e., the elements in $\mathbf{T_{t,t'}}$) while we are restricted by m equations only. Thus, the $m(m\text{-}1)$ degrees of freedom are sufficient to assure the solvability of the equation, except when $(\mathbf{q_t}| = \mathbf{0}$. This argument shows that a solution $\mathbf{T_{t,t'}}$ is not necessarily unique.

For example, if one of the initial apples is traded for one additional orange, consider encoding this in some matrix $\mathbf{T_{1,2}}$ such that multiplying $\mathbf{T_{1,2}}$ by the initial quantity yields the final quantity vector: $(\mathbf{q_2}| = (\mathbf{q_1}|\mathbf{T_{1,2}}$. A possible solution for $\mathbf{T_{1,2}}$ is

$$\mathbf{T_{1,2}} = \begin{bmatrix} -1 & -4 \\ 1 & 4 \end{bmatrix} \text{ because}$$

$$(\mathbf{q_2}| = (\mathbf{1, 1)} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & -4 \\ 1 & 4 \end{bmatrix} = (\mathbf{1}|\mathbf{T_{0,1}}\mathbf{T_{1,2}} = (\mathbf{1, 4}|$$

$$(\mathbf{q_2}| = (\mathbf{2, 3)} \begin{bmatrix} -1 & -4 \\ 1 & 4 \end{bmatrix} = (\mathbf{q_1}|\mathbf{T_{1,2}} = (\mathbf{1, 4}|$$

This example also illustrates how the associative property of matrix multiplication may be interpreted. A sequence of multiplications represents chronological transformations such that if $(\mathbf{q_t}|$ is equal to $(\mathbf{1}|\mathbf{T_{0,1}} \ldots \mathbf{T_{t\text{-}1,t}}$, then $(\mathbf{q_t}| = (\mathbf{1}|\mathbf{T_{0,t}}$ whereas $\mathbf{T_{0,t}} = \mathbf{\Pi}_1^{\,t}\ \mathbf{T}_{\tau\text{-}1,\tau}$. A series of transformations that take place within a time interval can be rendered by a single transformation defined over that time interval. This supports the representation of processes.

Continuing with trading one apple for one orange, three additional points are worth making. The first two impose restrictions on numbers in the transformation matrix; the third point relates to the fact that processes exist over a time interval.

Henceforth, I am banning any use of negative numbers. Axiomatically, all numbers to appear in the accounting measurement framework are posited to be positive or zero. The solution for $\mathbf{T_{1,2}}$ above is not valid. The preferred solution is

$$\mathbf{T_{1,2}} = \begin{bmatrix} ½ & ½ \\ 0 & 1 \end{bmatrix}, \text{ which indeed yields } (\mathbf{1, 4}| = (\mathbf{2, 3}| \begin{bmatrix} ½ & ½ \\ 0 & 1 \end{bmatrix}$$

This solution obtains from an analysis of what the process entails. Given the initial subset consisting of two apples, half of them are retained and half are sent away; and the whole subset consisting of three oranges is retained.

The requirement for positive numbers in the transformation matrices accounts for the fact that accounting deals with resources which are inherently positive quantities. This justifies the T structure of accounts prevailing with double-entry bookkeeping. This approach to constructing transformation matrices is generalizable to any dimension.

Further, resources in accounting are the outputs of processes and they arise from inputs. Resources exist as part of processes only. This has the following implications: The numbers in the transformation matrices are rational numbers between zero and one. The numbers in each of the rows add up to one. The numbers express rates of change, say the quantity of apples needed for one orange. As such, the numbers have dimensions, say units of orange divided by units of apple. Consequently, the linear transformation $\mathbf{T}_{t,t'}$ represents a multi-dimensional rate of change by means of which inputs are quantitatively transformed into outputs.

The third point worth making is a consequence of the solvability theorem. The argument used to establish the theorem also proves that there exists a $\mathbf{T}_{t',t}$, t < t', such that $(\mathbf{q}_t| = (\mathbf{q}_{t'}|\mathbf{T}_{t',t}$. The point is that some of the possible $\mathbf{T}_{t',t}$ may not be the inverse of any of the possible $\mathbf{T}_{t,t'}$.

When given a matrix **M**, its inverse $\mathbf{M}^{-1}$ is such that $\mathbf{M}\mathbf{M}^{-1} = \mathbf{I}$. For an inverse to exist, **M** must be a square matrix. Here, $\mathbf{T}_{t,t'}$ may not be square. Even when $\mathbf{T}_{t,t'}$ is square, it may not have an inverse in that mathematical sense. However, given inputs and outputs, both matrices $\mathbf{T}_{t,t'}$ and $\mathbf{T}_{t',t}$ always exist. This point introduces the final topic in this subsection.

The 'reversibility' of transformations is essential to the present formal framework. This is so not because accounting processes are reversible. Typically, they are not, say production processes. The reversal property embedded in the formalism is required because accounting processes exist out of the time dimension.

I claim that accounting deals with strategies. Strategies are defined as purposeful processes. As such, the representation of a strategy will account for (i) initial resources that are assumed as given; (ii) the strategy's final outcome that is associated with a purpose or goal chosen by a decision-maker; and (iii) the actual process, a sequence of resource transformations that connect the endowed inputs to the final outcome. The strategy will be interpreted as conceptually constructed by a decision-maker who projects expected transformations over time and the final outcome to a future time.

Thus, for t < t', $\mathbf{T}_{t,t'}$ is the causal matrix that encodes transformations in accordance with the flow of time, from past to future. The reversal matrix $\mathbf{T}_{t',t}$ is the teleological matrix that encodes transformations as they are conceived by the decision-maker working backwards along the time dimension, from future to past. This justifies the introduction of the covector P-space next.

Consider the covector space, which is isomorphic to $\mathbb{Q}^m$, and call it the P-space. I am denoting p-vectors by lower case bold letters within right semi-brackets, $|\mathbf{p})$. This purports to associate p-vectors with $m\times 1$ matrices, vertical lists of numbers. In particular, the p-vector $|\mathbf{1})$ will have all entries equal to 1.

$$|\mathbf{1}) \stackrel{\text{def}}{=} \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}_{m\times 1}$$

The solvability theorem holds in p-spaces. Replace transformation for revaluation, a term that does not entail a physical change and is consistent with decision-makers' projecting their views into the future. Denote a revaluation by $\mathbf{R}_{t,t'}$, t < t'.

Solvability theorem – $\forall\ |\mathbf{p}_{t'}) \neq 0,\ \forall\ |\mathbf{p}_t),\ t < t',\ \exists\ \mathbf{R}_{t,t'} : |\mathbf{p}_t) = \mathbf{R}_{t,t'}|\mathbf{p}_{t'})$

The representation of a strategy is based on the view that the causal transformations from past to future must be consistent with the teleological analysis that connects future outcomes to present means. This yields the equation, $(\mathbf{1}|\mathbf{TR}|\mathbf{1}) = 1$, which is interpreted as the accounting equation. From a mathematical perspective, this equation informs the existence of a tensor, which is constructed by multiplying physical transformations by conceptualised revaluations, whose measure is a dynamic constant over the time interval defined by the strategy and such that the constant is normalised to one.

2.2 Further comments on notation

Assume t = 1 and consider the Q_1 space that accounts for quantities at that time. For example, with only two apples the Q_1-space has one dimension and is therefore isomorphic to $\mathbb{Q}$. The applicable vector reads $(\mathbf{q_1}| = (\mathbf{apples}|[\mathbf{2}]$.

The symbol **[2]** conveys that this is a 1×1 matrix; **(apples|** is the one-dimensional vector in the basis that informs the quality of the resource defining this space.

The same logic applies to t = 2. For example, with only three oranges the applicable vector reads $(\mathbf{q_2}| = (\mathbf{oranges}|[\mathbf{3}]$. When dealing with dated quantity spaces, the dimensions of Q_t may vary. In this case, the transformations denoted by $\mathbf{T}_{t,t'}$ map earlier quantity spaces onto later ones.

The teleological approach to strategies require, however, a time-independent quantity space. This space is abstractly posited to exist without, therefore, any reference to a particular basis. This space is subjected to a direct sum-decomposition process that yields the dated subspaces.

For example, at t = 1 assume that the operator $\mathbf{T}_{0,1}$: Q → Q is a block diagonal one such that all entries are zero, except for $t_{1,1}$ which is set equal to 2; $[\mathbf{T}_{0,1}]_{11} = 2$. By imposing the condition that $(\mathbf{1}|\mathbf{T}_{0,1} = (\mathbf{q_1}|$, the Q-space is decomposed in a Q_1 subspace, which has $(\mathbf{q_1}|$ in its basis, and the complement thereof, $Q \ominus Q_1$, such that its basis remains undetermined. The same can be done in respect to t = 2, by reference to a $\mathbf{T}_{0,2}$ which sum-decomposes the space in an orange subspace and its complement. Since resources are now dated, the resulting subspaces intersect only in the null vector, $Q_1 \cap Q_2 = (\mathbf{0}|$.

This yields **T** as an assessment operator that is obtained from $\mathbf{T} = \mathbf{T}_{0,1} \oplus \ldots \oplus \mathbf{T}_{0,f}$. By reference to the foregoing theorem, **T** is diag($\mathbf{T}_{0,1}, \ldots, \mathbf{T}_{0,f}$), each blocked matrix $\mathbf{T}_{0,t}$ defining an invariant subspace of the Q-space. In matrix notation,

$$\mathbf{T_{0,1}} = \begin{bmatrix} [2] & 0 & \dots & 0 \\ 0 & \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \dots & & \dots \\ 0 & \dots & 0 \end{bmatrix}_{m-1\times m-1} \\ 0 & \\ \dots & \\ 0 & \end{bmatrix}_{m\times m} \qquad \mathbf{T_{0,2}} = \begin{bmatrix} [0] & 0 & 0 \dots 0 \\ 0 & [3] & 0 \dots 0 \\ 0 & 0 & \begin{bmatrix} 0 \dots 0 \\ \dots \\ 0 \dots 0 \end{bmatrix}_{m-2\times m-2} \\ \dots & & \\ 0 & 0 & \end{bmatrix}_{m\times m}$$

This procedure is such that invariant subspaces may account for more than one kind of resource. For example, suppose five bananas and one mango are identified to pertain to t = 3. Let $\mathbf{T_{0,3}}$ encode these resources. Since Q_3 is a bi-dimensional space, there are bananas and mangos, $\mathbf{T_{0,3}}$ is a 2×2 matrix acting on **(1|** at t = 3. The **T** that results from assessing resources at t = 1, 2, and 3, is shown next.

$$\mathbf{T} = \begin{bmatrix} [2] & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & [3] & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} & & 0 & \dots & 0 \\ & & & & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \begin{bmatrix} 0 & \dots & 0 \\ & \dots & 0 \\ 0 & \dots & 0 \end{bmatrix}_{m-4\times m-4} \\ \dots & & & & & & \\ 0 & 0 & 0 & 0 & & & \end{bmatrix}_{m\times m}$$

I rely on block diagonal matrices to represent resources. I rely on off-diagonal block matrices to represent transformations of resources. For example, assume $\mathbf{R_{f-2,f-1}}$ represents a revaluation within the interval [f-2, f-1) whereby the three-dimensional p-vector $|p_1, p_2, p_3\mathbf{)}$ defined at t = f-1 is re-expressed as the two-dimensional p-vector $|p_4, p_5\mathbf{)}$ defined at t = f-2. In matrix notation,

$$\mathbf{R_{f-1,f}} = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline r_{4,1} & r_{4,2} & r_{4,3} & 0 & 0 \\ r_{5,1} & r_{5,2} & r_{5,3} & 0 & 0 \end{array}\right]_{5\times 5}, \quad \mathbf{R_{f-1,f}} \in (P_{f-1} \oplus P_f) \text{ – the sum composition of the two dated P-spaces.}$$

The revaluation process takes the mathematical form of an equation, which is expressed by reference to matrix multiplication as follows.

$$\mathbf{R_{f-2,f-1}R_{f-1,f}|1)} = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline r_{4,1} & r_{4,2} & r_{4,3} & 0 & 0 \\ r_{5,1} & r_{5,2} & r_{5,3} & 0 & 0 \end{array}\right] \left[\begin{array}{ccc|cc} p_1 & 0 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 & 0 \\ 0 & 0 & p_3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ \hline 1 \\ 1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline p_4 \\ p_5 \end{array}\right]$$

All matrix multiplications will satisfy this block structure. This is because the bases chosen for both the quantity- and the p-spaces will be orthogonal, although not orthonormal. The resulting formalism frames accounting measurement as follows: resources are dated and pertain to strategies; the total value of resources associated with a particular strategy at a particular time can be calculated; given two moments in time, the difference in the values of resources is always equal to the net flow of values in and out of the strategy during the time interval defined by the two moments. Value within a strategy is axiomatically posited to be conserved. If so, the foregoing view is nothing other than the tautology that express the conservation of value.

3. Representational Theories of Measurement (RTM)

Since Chambers (1966) released his *Accounting, Evaluation and Economic Behaviour*, theorists have accepted the view that accounting measurement should either be summed in a manner consistent with RTM or be clearly shown inconsistent with RTM's tenets. In the latter case, an alternative approach would be required.

Measurement has been defined by Stevens (1946)[18] as the assignment of numerals to objects or events according to rules. This is where I commence discussing accounting measurement. Implicit in this approach is the view that numbers are not inherent to objects and events, but rather must be constructed—thus, the need for rules. Rules, in turn, are not expressed in the void; they require a context to gain meaning. The scope of RTM, as a theory of measurement, is to provide the context by means of mathematical structures, that is, by means of a set of mathematical objects, relations, operations, etc. that relate with one another in some logical way. An example of assigning numerals to objects is the weight of physical bodies. An example of assigning numerals to events is the perceived loudness of a sound.

RTM is based on the following epistemic claim: numbers assigned to objects or events convey meaning when shown to represent qualitatively observable features of reality. A two-steps procedure is required for the claim to hold. First, the observable features of reality are expressed within an appropriate mathematical framework. Roberts (1984: p.4) writes: "*We try to develop axioms or conditions under which measurement is possible*." Next, the axioms are tested by reference to actual observations. If empirically observed, the resulting framework is meaningful and mathematical statements therein will convey useful information about the objects of measurement (e.g., massive bodies) and the attribute being measured (e.g., weight).

The term 'axiom' is a misnomer. Axioms cannot be tested; they are syntactic in nature. However, this is the term for 'conditions of measurement' that prevails in the RTM literature. Subsuming a syntactic concept, axiom, within the empirically oriented terminology 'conditions of measurement' is unfortunate because it obscures the fact that RTM requires two very distinct, but complementary developments.

The development of axioms is an endeavour that requires mathematical skills and a conceptual mind. The development of empirical procedures, with the subsequent data collection and analysis, is a task to be completed by positive researchers. Empirical findings support or reject a theoretically given set of axioms and only then is a theory of measurement completed. The point is: introducing a mathematically consistent set of axioms is not enough to claim a valid theory of measurement.

The point—obvious as it may appear to RTM theorists—is particularly relevant in the case of accounting. All theorists that have ventured into accounting measurement (e.g., Chambers, 1966; Ijiri, 1967; Vickrey, 1970; Tippett, 1978) were focused on developing a framework wherein costs are additive. They "*...began to advocate accounting systems alternative to historic cost*" (Willett 1985: p.236) and when "*... show[ing] interest in questions of the*

[18] He was paraphrasing N.R. Campbell.

fundamental measurement of financial data, it is the issue of the additivity of accounting numbers which has dominated discussions.” (idem: p.235). When doing so, they failed to identify the qualitative features of accounting reality that should be observed. The reason is that accounting reality seems to arise directly in monetary terms. Accounting depends on invoices, which are financial objects bearing a cost or price.

Willett (1985, 1987, 1988, 1991) was acutely aware of this. Accordingly, he introduced the concept of a purely qualitative cost (e.g., Willett, 1987: p.165). Although he mentions that qualitative costs are observable without reference to a monetary standard, he does not show how to compare them without reference to money. He also pointed to yet another, related issue:

> *... In financial accounting the difficulty in identifying the attribute of interest lies not with the operation which assigns numbers (this can be done in a variety of ways such as historic cost, current cost, current purchasing power, net selling price etc.) but rather with the object or event possessing the attribute. (Willett 1985: p.41)*

The point is to show that the applicability of RTM to accounting measurement is not granted. Despite this cautionary comment, much is to be gained by working within RTM. The dichotomy between an observable world—wherein qualitative relationships may be observed—and a conceptual, representational framework—onto which these relationships are mapped and expressed quantitatively—is very useful and, I argue, essential in accounting. A tension is always present opposing the views of either endorsing or rejecting RTM.

1.1 Basic Facts

The origins of RTM can be traced back to two different traditions. As reported by Díez (1997), one tradition goes as far back as Helmoltz (1887) and Hölder (1901), and includes Campbell (1920); the other commences with Stevens (1946). Both traditions have been merged by Suppes’ (1951) introducing RTM as it has been later systematised in *Foundations of Measurement* (Krantz, Luce, Suppes, & Tversky, 1971; Suppes, Krantz, Luce, & Tversky, 1989; Luce, Krantz, Suppes, & Tversky, 1990).

The need to discuss axioms places RTM within the field of applied mathematics. As mentioned previously, axioms account for the first step in the development of an RTM approach to any particular branch of measurement. To provide a sense of what a RTM axiomatization entails, the case of extensive measurements is illustrated next. Weight or length are examples of extensive measurements.

In mathematics, when an arbitrary set A (the *underlying set*) is endowed with one or more operations it is called an algebraic structure. Examples of algebraic structures include groups and vector spaces. These structures are used in measurement theory wherein the underlying set A is endowed with relations and operations expressing purely qualitative properties. Such qualitative structure is called a *relational system*.

For example, consider (A, R, ∘) such that A contains the objects *a*, *b* and *c*, which may be placed on a two-pan balance. The qualitative relation R is interpreted such that *a*R*b* means “*a* is heavier than *b*”. The qualitative operation ∘ is interpreted such that *a*∘*b* means “*a* is combined with *b*”. Then (*a*∘*b*)R*c* means “*a* combined with *b* is heavier than *c*”.

Measurement theory addresses the issue of finding necessary and sufficient qualitative conditions, applicable to the objects in A, such that assigning numbers to these objects is "meaningful" (Roberts, 1984: ch.2).

Numbers are meaningfully assigned when the resulting numerical relations reflect the qualitative relations that hold among the objects in the underlying set. This is called the *representation problem*. The qualitative conditions are called the *axioms* for the representation.

When the numerical relations reflect the underlying, qualitative relations, then a homomorphism is said to exist between the relational system and its numerical representation. A homomorphism is a function *f* from the relational system onto the numerical system that preserves all the relations prevailing in the relational system. The function *f* is called a *scale*.

The relational system (A, R, ∘) above characterises what is known in measurement theory as an extensive structure.

Formally, extensive structures require that axioms be stated providing the necessary and sufficient conditions for the existence of a scale *f* that maps (A, R, ∘) onto (ℝ, >, +) and such that for all *a* and *b* in A the two following results are true.

(i) $a\mathrm{R}b$ iff $f(a) > f(b)$ – i.e., the scale preserves order;

(ii) $f(a \circ b) = f(a) + f(b)$ – i.e., the scale is additive.

A related issue is the *uniqueness problem* that addresses the possibility that many scales may exist, all of which would preserve the qualitative relations onto the numerical representation and satisfy (i) and (ii) above. As Roberts (1984: p.54) puts it, we can always perform mathematical operations on numbers (add them, average them, take logarithms, etc.) but the issue is whether, after performing such operations, we still end up with meaningful statements about the measured objects. This issue requires identifying a class of admissible transformations of scale in order to define a *scale type*. Common examples of scale types are the absolute, ratio, interval, ordinal, and nominal scale types.

The scale applicable to extensive structures is the ratio scale. We can always change the unit of mass by multiplying assigned numbers by a positive constant; thus, measuring mass in kilograms or in pounds is equally meaningful.

One set of necessary and sufficient axioms for the extensive scale *f* into (ℝ, >, +) is given by Hölder's theorem which states that an Archimedean ordered group provides an extensive relational system.

The Archimedean ordered group is defined by the following four axiomatic propositions:

P1. (A, ∘) is an algebraic group
P2. (A, R) is a strict simple order
P3. (Monotonicity) For all a, b, c in A; $a\mathrm{R}b$ iff $(a \circ c)\mathrm{R}(b \circ c)$ iff $(c \circ a)\mathrm{R}(c \circ b)$
P4. (Archimedean) For all a,b in A, if $a\mathrm{R}e$, where e is the identity for (A, ∘), then there is a positive integer n such that $na\mathrm{R}b$.

In summary, RTM requires scales that satisfy representational and uniqueness theorems. These scales have as their domain an empirical structure. This is the set containing the attribute-bearing objects of interest that satisfy certain relations and composition rules. The counter-domain or image of these scales is a numerical structure. The scales are viewed as the embedding, by means of the measurement procedure, of the empirical structure within the numerical structure.

1.2 Equivalence and partitions

RTM requires the objects in the empirical structure to be clearly identified along with the attribute being measured. For example, we identify massive objects and we may be interested in constructing a scale to express their weight. However, as Willett (1985) goes a long way to show, it is not obvious to identify the objects of accounting measurement and further to understand the nature to the accounting attribute.

I argue that the attribute-bearing objects are subsets of resources being transformed by some purposeful process, say production, trading, or a combination thereof. The input-output relationships define the attribute of value, that is, the value of the input is set equal to the output value. This key idea frames the concept of value in quite an original fashion.

This is because it implies the underlying R not being an order relation but rather an equivalence relation. Relations have properties. A well-known example is transitivity. A relation (A, R) is called transitive if, for all *a*, *b*, *c* in A, whenever *a*R*b* and *b*R*c*, then *a*R*c*. Further, several properties may apply to a single relation, say a relation (A, R) that is transitive, antisymmetric, and strongly complete is called a simple order.

The economic neoclassical concept of value requires consumers having preferences framed as relations that satisfy the properties of a simple order. Thus, the idea of order is paramount to the very definition of value in economics.

In contrast, my definition of value rests on the equivalence relation which satisfies the following properties: reflexivity, symmetry, and transitivity. For example, assume the relation 'has the same value as'. An apple relates to itself (*a*R*a* – reflexivity); given the apple being traded for a banana, the apple and the banana relate to each other (*a*R*b* and *b*R*a* – symmetry); given, further, the banana being traded for a carrot then *a*R*b* and *b*R*c* imply *a*R*c* – transitivity.

A set of elements in A such that they all relate to each other by means of an equivalence relation is called an equivalence class. The importance of equivalence for my purposes herein arises out of the following theorem:

> *Theorem 1.1 – Suppose (A,R) is an equivalence relation. Then:*
> *(a) Any two equivalence classes are either disjoint or identical;*
> *(b) The collection of (distinct) equivalence classes partitions A; that is, every element of A is in one and only one equivalence class. (Roberts, 1984, p.26)*

Statement (a) provides the basis for the accounting oriented classification scheme. Resources are given names (e.g., in-process inventory, non-current assets, etc.) according to how they enter as input in several, concomitant production processes. Further, the equivalence class to which they pertain contains future outputs associated with liabilities such that, by construction, the input-output equivalence relation entails the accounting equation.

Statement (b) provides the basis for *constructing* a partial order that will apply to the accounting values. Since the parts are contained in the whole, numbers being assigned to parts are smaller than those assigned to the whole and such that their sum equals the value of the whole. Since each part may at its turn be partitioned, yielding a finer partition, ever finer partitions yield numbers characterising a partial order.

I am relying on measurement theory to approach the concept of value with the following attitude:

> *Accept the fact that our preferences may not be rational or consistent in the usual utility-function sense. Much is to be gained by translating these preferences into concrete relations on numbers or other known mathematical objects, because then we get an accurate and understandable picture of our preferences … [Thus], rather than try to twist preferences into a given mold, report them in an insightful and useful way, and make the best of the information originally given you. (Roberts, 1972)*

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